Last time

- Proof of Serre
- Standard attaching maps
- Local case

Lemma 4 The following diagram commutes:

$$\begin{tikzcd}
E^1_{p,q} \arrow[r, equal, "\sim"] \arrow[d, "d^1"'] & C^{CW}_p(B;H_q(F)) \arrow[d, "d^{CW}"] \\
E^1_{p-1,q} \arrow[r, equal, "\sim"] & C^{CW}_{p-1}(B;H_q(F))
\end{tikzcd}$$

Proof From the community diagram

$$
\begin{tikzcd}
H_{p+q}(E_p, E_{p-1}) \arrow[r, "\delta"] \arrow[rr, bend left, "d^1_{pq}"] & H_{p+q-1}(E_{p-1}) \arrow[r, "j_*"] & H_{p+q-1}(E_{p-1}, E_{p-2}) \arrow[d, "\cong"] \\
\bigoplus_\alpha H_{p+q}(e_\alpha^* E, \partial e_\alpha^* E) \arrow[u, "\cong"] \arrow[r, "\delta"] & \bigoplus_\alpha H_{p+q-1}(\partial e_\alpha^* E) \arrow[u] & \tilde{H}_{p+q-1}\left(\bigvee_\beta e_\beta^* E / \partial e_\beta^* E\right) \arrow[d, "\cong"] \\
\bigoplus_\alpha H_{p+q}(D^p \times F, S^{p-1} \times F) \arrow[u, "\cong"] \arrow[r, "\delta"] & \bigoplus_\alpha H_{p+q-1}(S^{p-1} \times F) \arrow[u, "\cong"] & \bigoplus_\beta \tilde{H}_{p+q-1}\left(D^{p-1} \times F / S^{p-2} \times F\right) \arrow[d, "\cong"] \\
\bigoplus_\alpha H_q(F) \arrow[u, "\cong"] \arrow[r, equal] & \bigoplus_\alpha H_q(F) \arrow[u, "\cong"] \arrow[r, dashed, "(\star)"] & \bigoplus_\beta H_q(F),
\end{tikzcd}
$$

It suffices to determine the map $(*)$. Taking the attaching map for e_α to be standard with respect to e_β by Lemma 5, $(*)$ is determined by

$$\begin{tikzcd}
H_{p+q-1}(S^{p-1}\times F) \arrow[r] \arrow[d] & H_{p+q-1}(S^{p-1}\times F, S^{p-1}\setminus \bigcup \mathring{D}_i \times F) \arrow[d, "(**)"] \\
H_{p+q-1}(E_{p-1}, E_{p-2}) \arrow[r] & H_{p+q-1}(E_{p-1}, E_{p-1}\setminus \mathring{e}_\beta^* E)
\end{tikzcd},$$

which becomes

$$\bigoplus_i H_{p+q-1}(D_i\times F, \partial D_i\times F) \longrightarrow H_{p+q-1}(e_\beta^* E, \partial e_\beta^* E).$$

By Lemma 6 and method of local degrees, it suffices to show that the induced map on $H_*(F)$ is the identity. By construction, this map is the composite

$$F = F_b \xrightarrow{\cong} F_\alpha \xrightarrow{\cong} F_x = F_{g(x)} \xrightarrow{\cong} F_\beta \xrightarrow{\cong} F_b = F$$

of homotopy equivalences of the form L_γ for various paths γ. The composite is homotopic to the homotopy automorphism induced by the loop at b obtained by concatenating these paths, and $\pi_1(B,b)$ acts trivially on homology. $\square$

There is also a Serre spectral sequence in cohomology. Surprisingly, it is often much more useful.

<u>Thm</u> (Serre 3.0) Let $\pi: E \to B$ be a fibration with fiber F over a connected CW complex. If $\pi_1(B)$ acts trivially on $H_*(F)$, then there is a spectral sequence

$$E_2^{p,q} \cong H^p(B; H^q(F)) \Rightarrow H^{p+q}(E)$$

such that

(1) $d_r^{p,q} : E_r^{p,q} \longrightarrow E^{p+r,q-r+1}$

(2) (E_r, d_r) is a bigraded differential algebra, and

(3) the isomorphisms $E_{r+1} \cong H^*(E_r)$, $E_\infty \cong gH^*(E)$, and $E_2 \cong H^*(B; H^*(F))$ are rng isomorphisms.

Recall that (2) means that we have

associative and unital maps

$$E_r^{p_1,q_1} \otimes E_r^{p_2,q_2} \longrightarrow E_r^{p_1+p_2,\, q_1+q_2}$$

$$\alpha \otimes \beta \longmapsto \alpha\beta$$

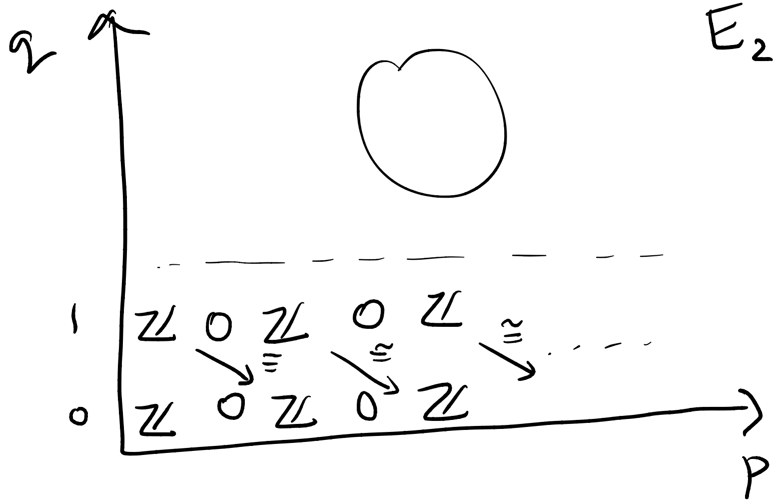
such that $\alpha\beta = (-1)^{(p_1+q_1)(p_2+q_2)} \beta\alpha$, and the Leibniz rule holds:

$$d_r(\alpha\beta) = (d_r\alpha)\beta + (-1)^{p_1+q_1} \alpha\, d_r\beta .$$

<u>Ex</u> Let us return to the example

$$S^1 \longrightarrow S^\infty \longrightarrow \mathbb{CP}^\infty .$$

As before, we must have the E_2-page



Writing $E_2^{0,1} = \langle a \rangle$ and $x := d_2 a$, we have $E_2^{2,0} = \langle x \rangle$, so $E_2^{2,1} = \langle xa \rangle$ under the isomorphism

$$H_p(\mathbb{CP}^\infty; H_q(S^1)) \cong H_p(\mathbb{CP}^\infty) \otimes H_q(S^1).$$

Thus, we have

$$d_2(xa) = d_2x \cdot a + (-1)^{2+0} x \cdot d_2 a$$

$$= x^2$$

$$\Longrightarrow E_2^{4,0} = \langle x^2 \rangle.$$

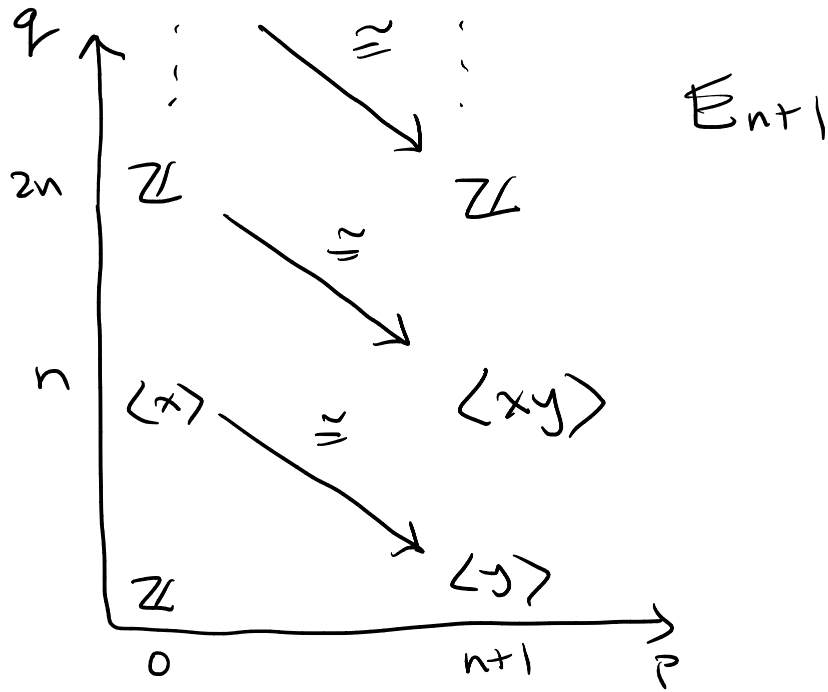
Continuing in this way, we find that

$$H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[x].$$

<u>Ex</u> Let us return to the example

$$\Omega S^{n+1} \longrightarrow PS^{n+1} \longrightarrow S^{n+1}.$$

As before, we have the E_{n+1}-page



Writing $E_{n+1}^{0,n} = \langle x \rangle$ and $y = d_{n+1}x$, we must ask: does x^2 generate $E_{n+1}^{0,2n}$? From the Leibniz rule, we have

$$
\begin{aligned}
d_{n+1} x^2 &= d_{n+1}x \cdot x + (-1)^n x \cdot d_{n+1} x \\
&= yx + (-1)^n xy \\
&= (-1)^{n(n+1)} xy + (-1)^n xy \\
&= (1 + (-1)^n) xy \\
&= \begin{cases} 2xy & n \text{ even} \\ 0 & n \text{ odd} \end{cases}
\end{aligned}
$$

Setting $x = x_1$ and $x_2 = d_{n+1}^{-1}(xy)$, it follows that $x_1^2 = \begin{cases} 2x_2 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$

Continuing in this way (exercise), we find that, for n even, $H^{kn}(\Omega S^{n+1})$ is infinite cyclic generated by x_k, and the ring structure is determined by the relations

$$x_k x_\ell = \binom{k+\ell}{k} x_{k+\ell}.$$

This ring is called the <u>divided power algebra</u> on one generator, written $\Gamma[x]$. For n odd,

$$H^*(\Omega S^{n+1}) \cong \Lambda[x] \otimes \Gamma[z], \qquad |x| = n, \ |z| = 2n.$$

We now accomplish our long sought after goal.

Thm $\pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z}$

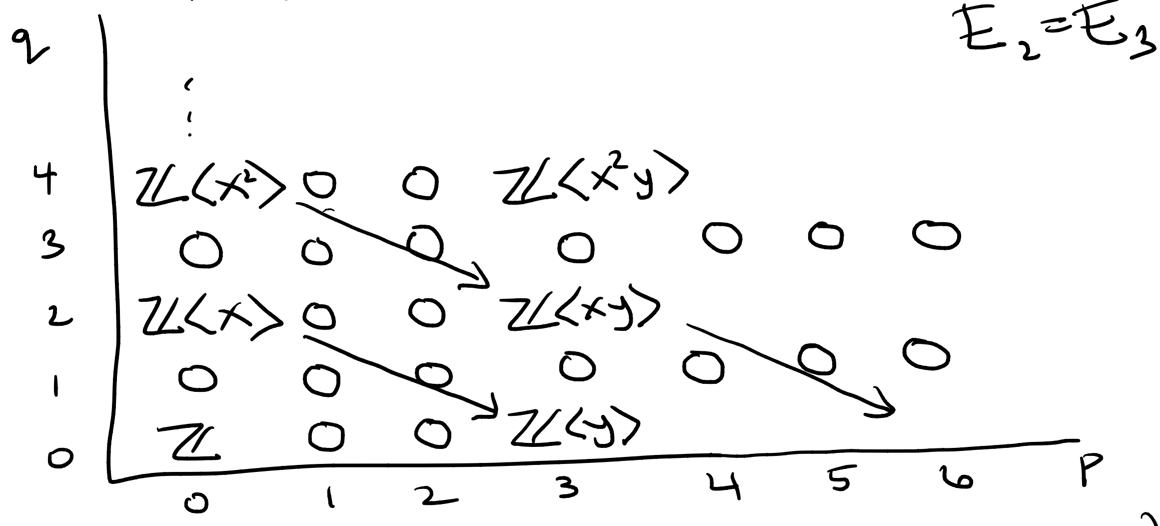
Lemma For $i \leq 6$, we have

$$H^i(K(\mathbb{Z},3)) \cong \begin{cases} \mathbb{Z} & i=0,3 \\ \mathbb{Z}/2\mathbb{Z} & i=6 \\ 0 & \text{otherwise}. \end{cases}$$

Proof Consider the SSS in cohomology for

$$\mathbb{CP}^\infty \simeq \Omega K(\mathbb{Z},3) \longrightarrow \cancelto{pt}{PK(\mathbb{Z},3)} \longrightarrow K(\mathbb{Z},3).$$

Then $E_\infty^{p,q} = 0$ for $(p,q) \neq (0,0)$, and Hurewicz and our knowledge of $\mathbb{CP}^\infty$ populate the E_2 page as follows:



For degree reasons, $d_2 \equiv 0$. To avoid contradiction at E_∞, we must have $H^4(K(\mathbb{Z},3)) = H^5(K(\mathbb{Z},3)) = 0$ and $d_3(x) = y$.

Then $d_3(x^2) = 2xy$ as before, so

$$y^2 = d_3(xy) \neq 0.$$

Since no further differentials enter or exit bidegree $(6,0)$, we have

$$H^6(K(\mathbb{Z},3)) \cong E_\infty^{6,0} \cong E_4^{6,0} \cong \mathbb{Z}/2\mathbb{Z}.$$

$\square$

<u>Proof of thm</u> Consider the SSS in cohomology for the fiber sequence

$$X \longrightarrow S^3 \xrightarrow{f} K(\mathbb{Z},3),$$

where $[f]$ generates $\pi_3 K(\mathbb{Z},3)$ and

$X = \mathrm{hofib}(f)$. The lemma and Hurewicz allow us to populate some of E_2:

$$
\begin{array}{c|ccccccc}
5 & H^5(X) & 0 & 0 & & 0 & 0 & \\
4 & H^4(X) & 0 & 0 & & 0 & 0 & \\
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 & 0 & \mathbb{Z}/2\mathbb{Z} \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
$$

For degree reasons, $d_2 \equiv d_3 \equiv d_4 \equiv d_5 \equiv 0$.
To avoid contradiction at E_∞, we must have $H^4(X) = 0$ and $d_6 : H^5(X) \xrightarrow{\cong} \mathbb{Z}/2\mathbb{Z}$. Thus,

$$H_4(X) \cong \mathbb{Z}^{\beta_4} \oplus T^{(?)} \cong \mathbb{Z}/2\mathbb{Z}.$$

Hence $\pi_4(X) \cong \mathbb{Z}/2\mathbb{Z}$ by Hurewicz, and the claim follows from the LES in homotopy.

$\square$

<u>Corollary</u> $\pi_{n+1} S^n \cong \mathbb{Z}/2\mathbb{Z}$ for $n > 2$. In particular, $\pi_1^s = \mathbb{Z}/2\mathbb{Z}$.