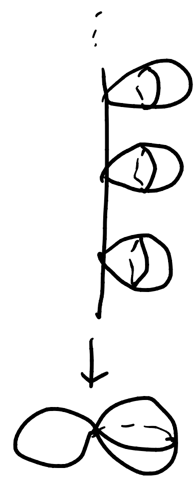
Last time

- Proof of Serre
- Multiplication in the SSS
- $\pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z}$

While this calculation may seem like cause for optimism, we should be cautious. Indeed, we don't even know homotopy groups of spheres are finitely generated!

Example The universal cover of $S^1 \vee S^2$ is $\mathbb{R} \coprod_{\mathbb{Z}} \mathbb{Z} \times S^2 \simeq \bigvee_{\mathbb{Z}} S^2$. By the LES in homotopy and Hurewicz,



$$\pi_2(S^1 \vee S^2) \cong \pi_2\Big(\bigvee_{\mathbb{Z}} S^2\Big)$$
$$\cong H_2\Big(\bigvee_{\mathbb{Z}} S^2\Big)$$
$$\cong \bigoplus_{\mathbb{Z}} H_2(S^2),$$

which is free Abelian of countably infinite rank.

The role of π_1 in this example is no coincidence.

<u>Thm</u> If X is simply connected, then $H_*(X)$ is of finite type iff $\pi_*(X)$ is so.

<u>Rmk</u> In fact, the conclusion holds under the weaker assumption that $\pi_1(X)$ is finite.

To understand how we might approach this result, let us ask the following.

Q Why is $\pi_4(S^3)$ finitely generated?

$\pi_4(S^3)$ f.g. $\iff \pi_4(X)$ f.g. (SSS)

$\iff H_4(X)$ f.g. (Hurewicz)

$\Longleftarrow$ $H_i(S^3)$ & $H_j(K(\mathbb{Z},3))$ f.g. (SSS)

Q Why is $H_i(K(\mathbb{Z},3))$ f.g.?

$H_i(K(\mathbb{Z},3))$ f.g. $\Longleftarrow H_k(\mathbb{C}P^\infty)$ f.g. (SSS)

$\Longleftarrow H_\ell(S^1)$ (SSS)

In short, we combine Hurewicz, Serre, and the fact that, given the exact sequence of Abelian groups

$$0 \to A \to B \to C \to 0,$$

B is f.g. iff A and C are so.

<u>Rmk</u> We heavily rely on the fact that $\mathbb{Z}$ is a <u>Noetherian</u> ring.

Def A **Serre class** is a collection $\mathcal{C} \neq \emptyset$ of Abelian groups such that, given the exact sequence

$$0 \to A \to B \to C \to 0,$$

$B \in \mathcal{C}$ iff $A, C \in \mathcal{C}$.

Exercise $\mathcal{C}$ is a Serre class iff $0 \in \mathcal{C}$, $\mathcal{C}$ is closed under isomorphisms, subobjects, quotients and extensions.

Ex (1) All Abelian groups

(2) Trivial groups

(3) Finitely generated Abelian groups

(4) Finite Abelian groups

(5) Torsion Abelian groups

(6) Abelian p-groups for p fixed

(7) Torsion Abelian groups without p-power torsion for p fixed

<u>Def</u> We say $\varphi : A \to B$ is a <u>$\mathcal{C}$-isomorphism</u> or an <u>isomorphism mod $\mathcal{C}$</u> if $\ker\varphi, \operatorname{coker}\varphi \in \mathcal{C}$. We say X is <u>$\mathcal{C}$-acyclic</u> if $H_n(X) \in \mathcal{C}$ for $n > 0$.

<u>Exercise</u> The relation of $\mathcal{C}$-isomorphism is an equivalence relation.

Lemma If $\mathcal{C}$ is a Serre class, then $\mathcal{C}$ is closed under finite direct sums.

Proof Induction using $0 \to A \to A \oplus B \to B \to 0$. $\square$

Lemma Given a finite filtration

$$0 = A_0 \leq A_1 \leq \cdots \leq A_n = A$$

of Abelian groups and a Serre class $\mathcal{C}$,

$A \in \mathcal{C}$ iff $\mathrm{gr}\, A \in \mathcal{C}$.

Proof If $A \in \mathcal{C}$, then $A_p, A_{p-1} \in \mathcal{C}$, so $\mathrm{gr}_p A = A_p / A_{p-1} \in \mathcal{C}$ for each p, so $\mathrm{gr}\, A \in \mathcal{C}$. Conversely, induction using $0 \to A_{p-1} \to A_p \to \mathrm{gr}_p A \to 0$ shows $A_p \in \mathcal{C}$ for every p. $\square$

Def A Serre class $\mathcal{C}$ is a Serre ring if $A \otimes B \in \mathcal{C}$ and $\mathrm{Tor}(A,B) \in \mathcal{C}$ for any $A, B \in \mathcal{C}$.

Ex The above examples are all Serre rings.

Reminders on Tor

(0) For Abelian groups A, B, $\mathrm{Tor}(A,B)$ is an Abelian group.

(1) $\mathrm{Tor}(A,B)$ is natural and symmetric in A, B and preserves direct sums.

(2) $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, A) \cong \ker(A \xrightarrow{n} A)$.

(3) Given the SES $0 \to A \to B \to C \to 0$, there is a LES

$$0 \to \mathrm{Tor}(A,D) \to \mathrm{Tor}(B,D) \to \mathrm{Tor}(C,D) \to A \otimes D \to B \otimes D \to C \otimes D \to 0.$$

(4) To calculate $\mathrm{Tor}(A,B)$, find a free resolution $0 \to F_1 \to F_0 \to A \to 0$. Then

$$\mathrm{Tor}(A,B) = \ker(F_1 \otimes B \to F_0 \otimes B).$$

(5) (UCT) There is a SES (unnaturally split)

$$0 \to H_n(X) \otimes A \to H_n(X;A) \to \mathrm{Tor}(H_{n-1}(X), A) \to 0$$

for any space X and Abelian group A.

<u>Exercise</u> Show that $\mathrm{Tor}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}$, where $d = \gcd(m,n)$.

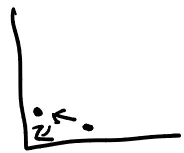
As foreshadowed above, Serre rings interact nicely with the Serre spectral sequence.

Prop Let $\mathcal{C}$ be a Serre ring and $F \to E \to B$ a fibration of path connected spaces such that $\pi_1(B)$ acts trivially on $H_*(F)$. If any two of F, E, B are $\mathcal{C}$-acyclic, then so is the third

Proof Suppose first that F and B are $\mathcal{C}$-acyclic. By the UCT, $E^2_{p,q} \in \mathcal{C}$ for every $(p,q) \neq (0,0)$. By induction, $E^r_{p,q} \in \mathcal{C}$, so $E^r_{p,q} \geq \ker d^r, \operatorname{im} d^r \in \mathcal{C}$, so $E^{r+1}_{p,q} \cong \ker d^r / \operatorname{im} d^r \in \mathcal{C}$. Thus, $E^\infty_{p,q} \in \mathcal{C}$ for $(p,q) \neq (0,0)$, so for $n > 0$

$$\operatorname{gr} H_n(E) \cong \bigoplus_{p+q=n} E^\infty_{p,q} \in \mathcal{C} \Rightarrow H_n(E) \in \mathcal{C}.$$

Suppose instead that E and B are $\mathcal{C}$-acyclic. Then $\operatorname{gr} H_n(E) \in \mathcal{C}$ for $n>0$, so $E^\infty_{pq} \in \mathcal{C}$ for $(p,q) \neq (0,0)$. We will show by induction that $H_q(F) \in \mathcal{C}$ for $q>0$. For the base case $q=1$, we have

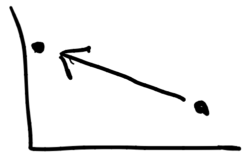


$$\begin{tikzcd}
 & 0 & & & \\
0 \arrow[r] & \operatorname{im} d^2 \arrow[u] \arrow[r] & E^2_{0,1} \arrow[r] & E^\infty_{0,1} \arrow[r] & 0. \\
 & E^2_{2,0} \arrow[u] \arrow[ur, "d^2"'] & & &
\end{tikzcd}$$

Since $E^2_{2,0} = H_2(B) \in \mathcal{C}$, $\operatorname{im} d^2 \in \mathcal{C}$. Since $E^\infty_{0,1} \in \mathcal{C}$, $H_1(F) = E^2_{0,1} \in \mathcal{C}$, as desired. For the induction step, assuming $H_q(F) \in \mathcal{C}$ for $0<q<k$, the

same argument as above shows that $E^r_{p,q} \in \mathcal{C}$ for every $(p,q) \neq (0,0)$ with $q < k$.

Consider the exact sequence



$$\begin{tikzcd}
 & 0 & & & \\
0 \arrow[r] & \operatorname{im} d^r \arrow[u] \arrow[r] & E^r_{0,k} \arrow[r] & E^{r+1}_{0,k} \arrow[r] & 0 \\
 & E^r_{r,k-r+1} \arrow[u] & & &
\end{tikzcd}$$

Since $k-r+1 < k$, $E^r_{r,k-r+1} \in \mathcal{C}$, so $\operatorname{im} d^r \in \mathcal{C}$, so $E^r_{0,k} \in \mathcal{C}$ iff $E^{r+1}_{0,k} \in \mathcal{C}$. Since containment holds for r sufficiently large, we have $H_k(F) = E^2_{0,k} \in \mathcal{C}$ by downward induction.

The remaining case is an exercise. $\square$

Cor Let $\mathcal{C}$ be a Serre ring. If X is simply connected, then X is $\mathcal{C}$-acyclic iff ΩX is so.

Def A Serre class $\mathcal{C}$ is *acyclic* if $K(A,1)$ is acyclic for $A \in \mathcal{C}$.

Cor If $\mathcal{C}$ is an acyclic Serre ring, then $K(A,n)$ is $\mathcal{C}$-acyclic for $A \in \mathcal{C}$ and $n > 0$.

Prop Examples (1)–(7) above are acyclic.

We defer this result for now.

Our desired finite generation result is a special case of the following.

<u>Thm</u> Let X be a simply connected space. For any acyclic Serre ring $\mathcal{C}$, X is $\mathcal{C}$-acyclic iff $\pi_i(X) \in \mathcal{C}$ for $i \geq 0$.

The result follows from a version of Hurewicz.

<u>Thm</u> (Mod $\mathcal{C}$ Hurewicz) Let $\mathcal{C}$ be an acyclic Serre ring. If X is simply connected and $\pi_i(X) \in \mathcal{C}$ for $i < n$, then the Hurewicz homomorphism $\pi_n(X) \to H_n(X)$ is a $\mathcal{C}$-isomorphism.