Last time

- Serre class theory
- $\mathcal{C}$-acyclicity and SSS
- Mod $\mathcal{C}$ Hurewicz

The result follows from a version of Hurewicz.

Thm (Mod $\mathcal{C}$ Hurewicz) Let $\mathcal{C}$ be an acyclic Serre ring. If X is simply connected and $\pi_i(X) \in \mathcal{C}$ for $i < n$, then the Hurewicz homomorphism $\pi_n(X) \to H_n(X)$ is a $\mathcal{C}$-isomorphism.

Thm Let X be a simply connected space. For any acyclic Serre ring $\mathcal{C}$, X is $\mathcal{C}$-acyclic iff $\pi_i(X) \in \mathcal{C}$ for $i \geq 0$.

This result follows from Mod $\mathcal{C}$ Hurewicz and the following easy lemma.

Lemma If $A \cong B \bmod \mathcal{C}$, then $A \in \mathcal{C}$ iff $B \in \mathcal{C}$.

Proof Given a homomorphism $\varphi : A \longrightarrow B$ with $\ker\varphi, \operatorname{coker}\varphi \in \mathcal{C}$, consider

$$0 \to A/\ker\varphi \to B \to \operatorname{coker}\varphi \to 0$$

$$0 \to \ker\varphi \to A \to \operatorname{im}\varphi \to 0 \qquad \square$$

Proof of thm: If $\pi_i(X) \in \mathcal{C}$ for every i, then $\pi_i(X) \cong H_i(X)$ mod $\mathcal{C}$ by Hurewicz, so X is $\mathcal{C}$-acyclic by the lemma.

Conversely, if X is $\mathcal{C}$ acyclic, let n be minimal number such that $\pi_n(X) \notin \mathcal{C}$. Then $\pi_n(X) \cong H_n(X)$ mod $\mathcal{C}$ by Hurewicz, so $\pi_n(X) \in \mathcal{C}$ by the lemma, a contradiction.

$\square$

We now turn to the proof of mod $\mathcal{C}$ Hurewicz.

Idea We understand $\pi_4(S^3)$ as the first nonzero homotopy group of a space X with the same higher homotopy groups, which differs from S^3 by an Eilenberg-MacLane space.

Thm ("Whitehead tower") Let X be path connected. There exists a sequence of maps

$$\cdots \longrightarrow X^{(1)} \longrightarrow X^{(0)} = X$$

such that

(1) $X^{(n)}$ is n-connected

(2) $\pi_i(X^{(n)}) \xrightarrow{\cong} \pi_i(X)$ for $i > n$.

__Lemma__ If $\cdots \to X^{(1)} \to X^{(0)} = X$ is a Whitehead tower for X, then

$$\operatorname{hofiber}\left(X^{(n)} \to X^{(n-1)}\right) \simeq K(\pi_n(X), n-1).$$

__Proof__ Write F_n for the homotopy fiber.

$$\cdots \to \pi_i(F_n) \to \pi_i(X^{(n)}) \xrightarrow{p^{(n)}_*} \pi_i(X^{(n-1)}) \to \pi_{i-1}(F_n) \to \cdots$$

For $i \neq n$, $p^{(n)}_*$ is an isomorphism, so

$\pi_i(F_n) = 0$ for $i \neq n, n-1$, in which case we have

$$\begin{tikzcd}
0 \arrow[r] & \pi_n(F_n) \arrow[r] & \cancelto{0}{\pi_n(X^{(n)})} \arrow[r] & \cancelto{\pi_n(X)}{\pi_n(X^{(n-1)})} \arrow[r] & \pi_{n-1}(F_n) \arrow[d] \\
 & & & & \cancelto{0}{\pi_{n-1}(X^{(n)})}
\end{tikzcd}$$

. $\square$

We deduce this "truncation from below" from a corresponding "truncation from above"!

<u>Thm</u> ("Postnikov tower") Let X be a path connected space. There is a commutative diagram

$$\begin{tikzcd} & \vdots \arrow[d] \\ & X_{(2)} \arrow[d] \\ X \arrow[r] \arrow[ur] \arrow[uur] & X_{(1)} \end{tikzcd}$$

such that

(1) $\pi_i(X_{(n)}) = 0$ for $i > n$

(2) $\pi_i(X) \xrightarrow{\cong} \pi_i(X_{(n)})$ for $i \leq n$.

Proof of both We assume that X is a CW complex. For Postnikov, construct $X_{(n)}$ from X by attaching cells of dimension $> n+1$ to kill higher homotopy groups.

By the extension lemma, the dashed filler exists so the commutative diagrams

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_{(n)} \\ X_{(n+1)} \arrow[ur, dashed] & \end{tikzcd}$$

For Whitehead, define
$$X^{(n)} = \mathrm{hofiber}\,(X \to X_{(n)})$$
and consider the LES in homotopy.

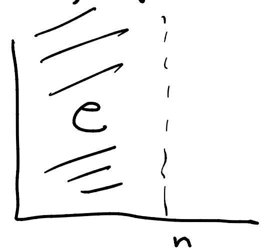
$\square$

<u>Proof of Mod $\mathcal{C}$ Hurewicz</u> Consider a Whitehead tower for X

$$\cdots \longrightarrow X^{(1)} \to X^{(0)} \longrightarrow X.$$

WLOG, $X^{(n)} \xrightarrow{p(n)} X^{(n-1)}$ is a fibration with fiber $F_n \simeq K(\pi_n(X), n-1)$ by the lemma. Since $\mathcal{C}$ is acyclic and $\pi_i(X) \in \mathcal{C}$ for every $i<n$, we conclude that F_i is $\mathcal{C}$-acyclic

for every $i<n$. By induction on n, we also know that $H_i(X) \in \mathcal{C}$ for $i<n$. Applying the argument of the 2-out-of-3 proposition to the situation



$$F_2 \longrightarrow X^{(2)} \longrightarrow X^{(1)} = X,$$

it follows that $H_n(X) = H_n(X^{(2)}) \bmod \mathcal{C}$.

Continuing in this way, $H_n(X) \cong H_n(X^{(n-1)}) \bmod \mathcal{C}$. Since, in the commuting diagram

$$\begin{tikzcd}
\pi_n(X^{(n-1)}) \arrow[r, "\cong"] \arrow[d, "\cong"'] & \pi_n(X) \arrow[d] \\
H_n(X^{(n-1)}) \arrow[r, "\cong_{\mathcal{C}}"] & H_n(X)
\end{tikzcd}$$

the lefthand arrow is an isomorphism by Hurewitz, and the top by construction, the claim follows.

$\square$

<u>Prop</u> The following Serre rigs are acyclic:

(1) all
(2) trivial
(3) f.g.
(4) finite
(5) torsion
(6) p-groups
(7) prime-to-p torsion

Lemma $K(A\times B, n) \simeq K(A,n)\times K(B,n)$.

Proof Exercise. $\square$

Lemma Let $\{A_i\}$ be the set of finitely generated subgroups of A, ordered by inclusion.

(1) $\{A_i\}$ is a directed system

(2) $A \cong \varinjlim A_i$

(3) $H_n(K(A,1)) \cong \varinjlim H_n(K(A_i,1))$.

Proof Claims (1) and (2) are exercises.

For (3), take $K(A,1)$ to be the CW complex with $K(A,1)_1 = \bigvee_A S^1$ and defining $K(A,1)_n$ by attaching n-cells along <u>every</u> pointed map $S^{n-1} \to K(A,1)_{n-1}$. For $S \subseteq A$, let $X_S \subseteq K(A,1)$ be the largest subcomplex containing $\bigvee_S S^1 \subseteq K(A,1)_1$. Then (exercise) $X_S \simeq K(\langle S \rangle, 1)$, and any compact subspace $K \subseteq K(A,1)$ lies in X_S for some <u>finite</u> subset $S \subseteq A$, since it lies in a finite subcomplex of $K(A,1)$. Since simplices are compact, the usual argument applies. $\square$

<u>Prop</u> We have

$$\tilde{H}_i(K(C_n,1)) \cong \begin{cases} \mathbb{Z}/n\mathbb{Z} & i \text{ odd} \\ 0 & i \text{ even} \end{cases}$$

<u>Rmk</u> This calculation can be done using cellular homology, as for $\mathbb{R}P^\infty$.

<u>Construction</u> The group C_n acts on $\mathbb{C}^r$ by $e^{2\pi i/n}$ in each coordinate. This action is

compatible with the inclusions $\mathbb{C}^r \subseteq \mathbb{C}^{r+1}$ and preserves $S^{2r-1} \subseteq \mathbb{C}^r$, so it defines an action on S^∞.

<u>Prop</u> $S^\infty/C_n \simeq K(C_n, 1)$.

<u>Proof</u> The projection $S^\infty \longrightarrow \mathbb{C}P^\infty$ is C_n-equivariant, where C_n acts trivially on $\mathbb{C}P^\infty$. Thus, we have a fiber bundle

$$S^1/C_n \longrightarrow S^\infty/C_n \longrightarrow \mathbb{C}P^\infty.$$

Clearly, $S^1/C_n \cong S^1$, and the LES in homotopy for $C_n \to S^\infty \to S^\infty/C_n$ shows that S^∞/C_n is a $K(\pi, 1)$. From the diagram

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d] & S^\infty \arrow[r] \arrow[d] & \mathbb{CP}^\infty \arrow[d, equal] \\ S^1/C_n \arrow[r] & S^\infty/C_n \arrow[r] & \mathbb{CP}^\infty \end{tikzcd}$$

we obtain

$$\begin{tikzcd} 0 \arrow[r] & \pi_2(\mathbb{CP}^\infty) \arrow[r, "\cong"] \arrow[d, equal] & \pi_1(S^1) \arrow[r] \arrow[d, "n"] & \cancel{\pi_1(S^\infty)} \overset{0}{} \arrow[d] & \\ 0 \arrow[r] & \pi_2(\mathbb{CP}^\infty) \arrow[r] & \pi_1(S^1/C_n) \arrow[r] & \pi_1(S^\infty/C_n) \arrow[r] & 0, \end{tikzcd}$$

so $\pi_1(S^\infty/C_n) \cong C_n$. $\square$