Last time

- Whitehead + Postnikov towers
- Proof of Mod $\mathcal{C}$ Hurewicz
- Horizontal Iwasawa theory
- $a^n + b^n = c^n \Rightarrow n > 2$

<u>Prop</u> The following Serre rigs are acyclic:

(1) all

(2) trivial

(3) f.g.

(4) finite

(5) torsion

(6) p-groups

(7) prime-to-p torsion

Prop We have

$$\tilde{H}_i(K(C_n,1)) \cong \begin{cases} \mathbb{Z}/4\mathbb{Z} & i \text{ odd} \\ 0 & i \text{ even} \end{cases}$$

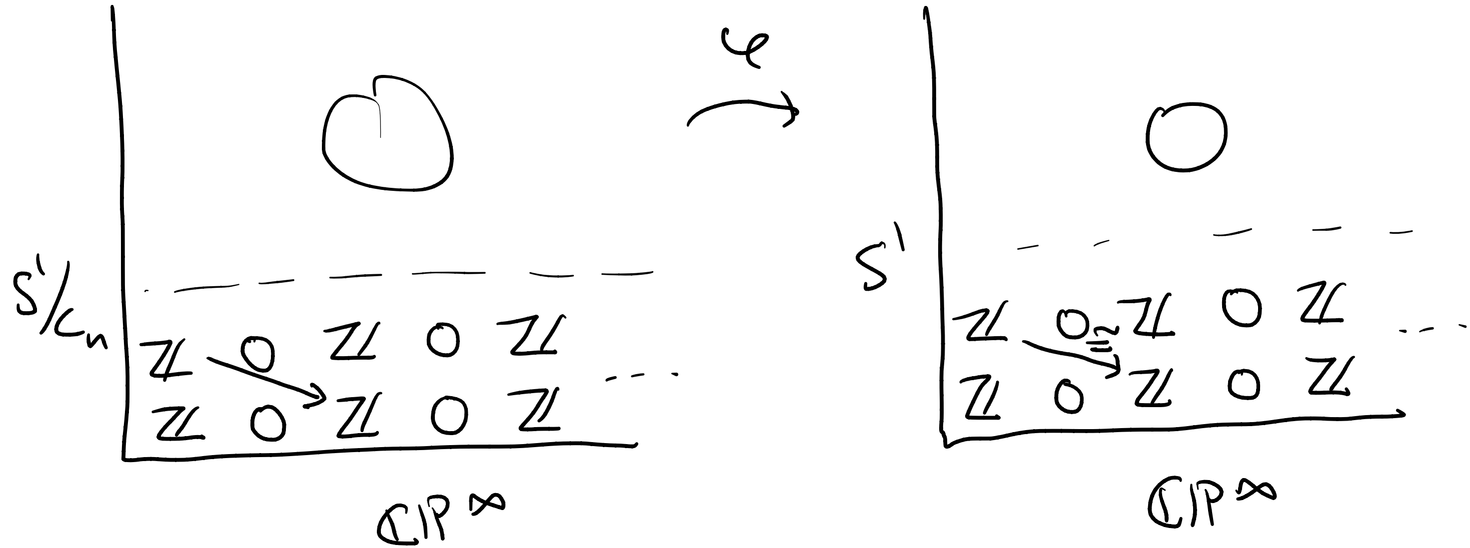
Proof of acyclicity For (1), there is nothing to show. For (2), $K(0,1)$ is modeled by a singleton. For (3), by the classification of finitely generated Abelian groups, Künneth, and closure under $\oplus$, $\otimes$, and Tor, it suffices to observe that $K(\mathbb{Z},1)$ and $K(C_n,1)$ for every n have homology of finite type. For (4), for the same reason, it suffices to observe that $H^*(K(C_n,1))$ is finite in each positive degree. For (5),

since a direct limit of torsion groups is torsion, the same reasoning applies. For (6) and (7), we instead observe that $H^*(K(C_n,1))$ in each degree is torsion of order dividing n. $\square$

Proof of prop From the commutative diagram

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d] & S^\infty \arrow[r] \arrow[d] & \mathbb{CP}^\infty \arrow[d, equal] \\ S^1/C_n \arrow[r] & S^\infty/C_n \arrow[r] & \mathbb{CP}^\infty \end{tikzcd}$$

we obtain a map of cohomology SSS's:



Since $\varphi_{0,1}$ is multiplication by n and UCT, it follows that the lefthand $d_2^{0,1}$ is also multiplication by n. Using the multiplicative structure as before, it follows that every lefthand d_2 component is multiplication by n, so

$$\tilde{H}^i(K(C_n,1)) \cong \begin{cases} \mathbb{Z}/n\mathbb{Z} & i>0 \text{ even} \\ 0 & i \text{ odd} \end{cases}.$$

The claim then follows from UCT (we use that $H_*(K(C_n,1))$ is finite type, since $H_*(S^1/C_n)$ and $H_*(\mathbb{C}P^\infty)$ are so).

$\square$

Thus, homotopy groups of spheres are finitely generated, so their calculation amounts to determining ranks and p-primary components. The rank problem has a complete answer.

Thm (Serre)

$$\operatorname{rk} \pi_i(S^n) = \begin{cases} 1 & i=n \text{ or } i=2n-1,\ n \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

In particular, the higher homotopy groups of an odd sphere are finite, and the same holds for an even sphere with a single exception.

Cor For $i>0$, π_i^s is finite.

Rmk So far, we have $\pi_0^s \cong \mathbb{Z}$, $\pi_1^s \cong \mathbb{Z}/2\mathbb{Z}$.

Given this, we will calculate π_2^s. A closed form answer for π_i^s in general is not known (and not expected).

Strategy Work mod the Serre class of torsion groups.

Lemma A homomorphism $\varphi: A \to B$ is an isomorphism mod torsion iff $\varphi \otimes \mathbb{Q}$ is an isomorphism.

Proof Consider the exact sequence

$$0 \to \ker \varphi \to A \to B \to \operatorname{coker} \varphi \to 0.$$

From the theory of localizations of rings and modules, $\mathbb{Q}$ is a flat $\mathbb{Z}$-module, so it suffices to show that $C \otimes \mathbb{Q} = 0$ iff C is torsion. For the "if" direction, given a pure tensor $c \otimes q$ with $nc = 0$, we have $c \otimes q = nc \otimes \frac{q}{n} = 0$. Since pure tensors span $C \otimes \mathbb{Q}$, the claim follows. For the converse, consider the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{c} C,$$

where $|c| = \infty$. $\square$

<u>Observation</u> The canonical map $S^n \to K(\mathbb{Z},n)$ is an isomorphism on π_n, hence on H_n by Hurewicz, hence on $H_n(-;\mathbb{Q}) \cong H_n \otimes \mathbb{Q}$ by UCT.

<u>Prop</u> For $n>0$,

$$H^*(K(\mathbb{Z},n);\mathbb{Q}) \cong \begin{cases} P[x] & n \text{ even} \\ \Lambda[x] & n \text{ odd} \end{cases}$$

where $|x|=n$.

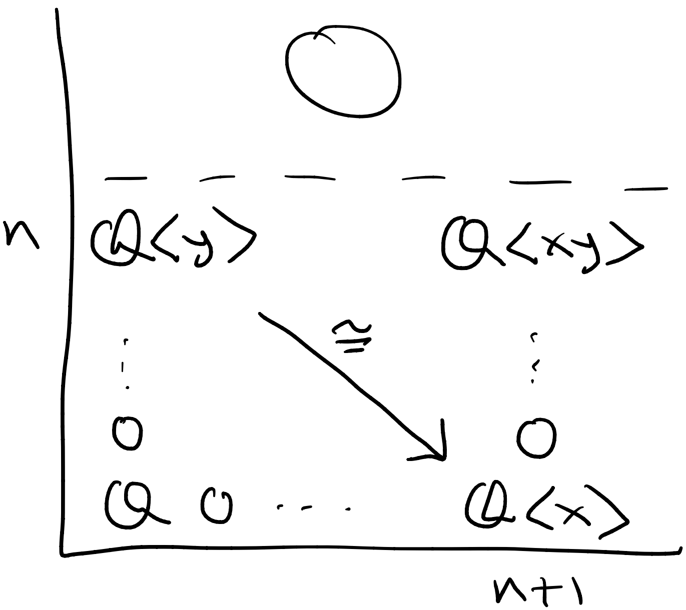
<u>Proof</u> We proceed by induction on n, the case of $K(\mathbb{Z},1) \simeq S^1$ being known. Consider the SSS in cohomology for the fibration

$$\Omega K(\mathbb{Z},n+1) \longrightarrow PK(\mathbb{Z},n+1) \longrightarrow K(\mathbb{Z},n+1)$$

$PK(\mathbb{Z},n+1) \simeq pt$

$$\Omega K(\mathbb{Z},n+1) \cong K(\mathbb{Z},n)$$

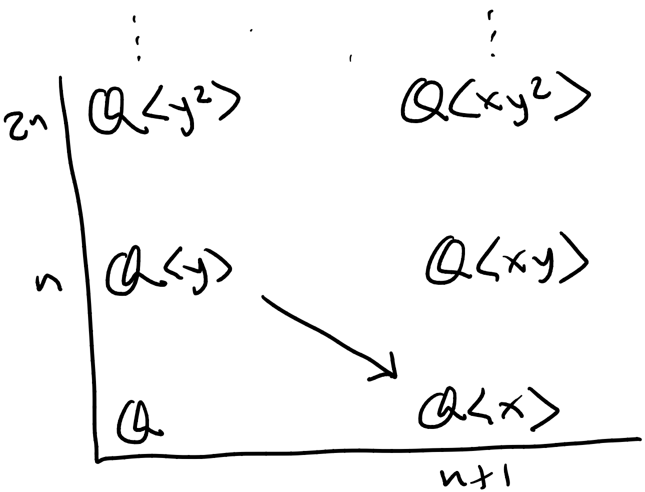
and assume first that n is odd. The induction hypothesis and Hurewicz give complete information on E_2 for $p \le n+1$



For degree reasons, $d_r \equiv 0$ for $r \neq n+1$. Since $E_\infty^{p,q} = 0$ for $(p,q) \neq (0,0)$, $d_{n+1}^{0,n}$ is an isomorphism, say $d_{n+1} y = x$; whence

$d_{n+1} xy = (-1)^{n+1} x^2$. To avoid contradiction, we must have $E_2^{p,0} = 0$ for $n+1 < p < 2n+2$, $x^2 \neq 0$, and $E_2^{2n+2,0} = \mathbb{Q}\langle x^2 \rangle$. We now repeat this analysis inductively after shifting to the right by $n+1$.

Assume now that n is even. The same reasoning applies to show that $d_{n+1} y = x$ is an isomorphism. By Leibniz, since $|y| = n$ is even, we have the



formula $d_{n+1} y^r = r x y^{r-1}$ (exercise, induction) which is also an isomorphism, since we work over $\mathbb{Q}$. To avoid contradiction at E_∞, we must have $E_2^{p,q} = 0$ for $p > n+1$.

$\square$