Last time

- $H_*(K(C_n,1))$
- Acyclicity
- Serre's theorem on $\mathrm{rk}\,\pi_i(S^n)$
- $H^*(K(\mathbb{Z},n);\mathbb{Q})$

We require one more concept from Serre class theory.

<u>Def</u> A Serre class $\mathcal{C}$ is a <u>Serre ideal</u> if $A \otimes B \in \mathcal{C}$ and $\mathrm{Tor}(A,B) \in \mathcal{C}$ for $A \in \mathcal{C}$ and B arbitrary.

Ex The Serre rings of finitely generated and finite Abelian groups are not Serre ideals. Our other examples are.

A relative mod $\mathcal{C}$ Hurewicz theorem holds for Serre ideals (proof omitted). Via mapping cylinders, it implies a "Whitehead" theorem.

Cor If $\mathcal{C}$ is an acyclic Serre ideal and $f: X \to Y$ a map between simply connected spaces, then f induces an isomorphism mod $\mathcal{C}$ on homotopy groups iff it does so on homology.

Proof of thm, n odd consider the map

$$S^n \xrightarrow{f} K(\mathbb{Z},n).$$

We have already observed that f induces an isomorphism on $H_n(-;\mathbb{Q})$ hence on $H^n(-;\mathbb{Q})$, hence on $H^*(-;\mathbb{Q})$ by the calculation, since n is odd, hence on $H_*(-;\mathbb{Q}) \cong H_* \otimes \mathbb{Q}$. We conclude that f induces an isomorphism mod torsion on homology, hence on homotopy groups. Since $\pi_i(K(\mathbb{Z},n)) = 0$ for $i > n$, it follows that $\pi_i(S^n)$ is torsion for $i > n$, hence finite. $\square$

We aim to reduce the even case to the odd case. Consider the unit tangent bundle of S^n, i.e.,

$$E = \{ (x_1, x_2) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid x_1 \cdot x_2 = 0, |x_i| = 1 \}.$$

The projection $E \longrightarrow S^n$ onto the first factor is a fiber bundle with fiber S^{n-1}.

Note that homotopy / homology groups of E are finitely generated from the LES.

<u>Lemma</u> $\pi_{n-1}(E)$ is finite.

<u>Proof</u> From the LES

$$\pi_n(E) \xrightarrow{p_*} \pi_n(S^n) \xrightarrow{\partial} \pi_{n-1}(S^{n-1}) \longrightarrow \pi_{n-1}(E) \to \cancelto{0}{\pi_{n-1}(S^n)}$$

It suffices to show that $p_* : \pi_n(E) \to \pi_n(S^n)$ is not surjective, but n is even, so p admits no section. $\square$

<u>Lemma</u> $\tilde{H}_i(E;\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & i = 2n-1 \\ 0 & \text{otherwise} \end{cases}$

<u>Proof</u> Consider the SSS for $E \to S^n$:

$$\begin{array}{c|ccc} n-1 & \mathbb{Q} & & \mathbb{Q} \\ 0 & \mathbb{Q} & & \mathbb{Q} \\ \hline & 0 & & n \end{array}$$

The only possible nonzero differential is $d^n_{n,0}$, so it suffices to show that $H_{n-1}(E;\mathbb{Q}) = 0$. If not, then $H_{n-1}(E)$ has positive rank by UCT, so $\pi_{n-1}(E)$ is infinite by Hurewicz, a contradiction. $\square$

<u>Proof of thm, n even</u> By the lemma and mod $\mathcal{C}$ Hurewicz, $\pi_i(E)$ is torsion for $i < 2n-1$, and $\pi_{2n-1}(E)$ has rank 1. Representing a generator of a $\mathbb{Z}$ summand by $S^{2n-1} \xrightarrow{g} E$, we see that g induces an isomorphism on rational homology,

hence an isomorphism mod torsion on homotopy groups. The claim follows from the LES

$$\cdots \to \pi_i(S^{n-1}) \to \pi_i(E) \to \pi_i(S^n) \to \cdots$$

and the odd case for $n-1$ and $2n-1$.

$\square$

Q What can we say about p-primary components?

In view of Freudenthal, it is reasonable to begin with S^3. Write A_p for the p-primary component of the finite Abelian group A.

Thm For any prime p,

$$\pi_i(S^3)_p \cong \begin{cases} \mathbb{Z}/p\mathbb{Z} & i=2p \\ 0 & 3<i<2p \end{cases}.$$

Lemma Let $X = \mathrm{hofib}(S^3 \to K(\mathbb{Z},3))$. We have

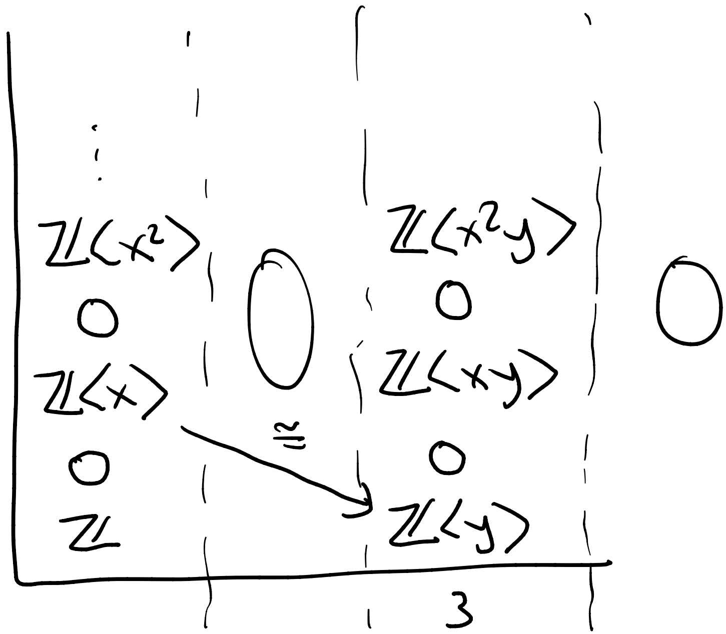
$$\tilde{H}_k(X) \cong \begin{cases} \mathbb{Z}/n\mathbb{Z} & k=2n \\ 0 & \text{otherwise} \end{cases}.$$

Proof From the LES in homotopy,

$$\mathrm{hofib}(X \to S^3) \simeq K(\mathbb{Z},2) \simeq \mathbb{C}P^\infty,$$

so consider cohomology SSS for

$$\mathbb{CP}^\infty \to X \to S^3.$$



For degree reasons, d_3 is the only possible nonzero differential. Since X is 3-connected, $d_3 x = y$ by Hurewicz.

By Leibniz, since $|x|$ is even, $d_3 x^n = n x^{n-1} y$.

Since $E_4 = E_\infty$, the claim follows from UCT.

$\square$

We write $\mathcal{C}_p$ for the class of prime-to-p torsion Abelian groups.

<u>Proof of thm</u> By the lemma, $H_i(X) \in \mathcal{C}_p$ for $0 < i < 2p$, so mod $\mathcal{C}_p$ Hurewicz implies that $\pi_i(X) \in \mathcal{C}_p$ in this range, and

$$\pi_{2p}(X) \cong H_{2p}(X) \cong \mathbb{Z}/p\mathbb{Z} \mod \mathcal{C}_p.$$

Since $\pi_i(X) \cong \pi_i(S^3)$ for $i > 3$, and since $A \cong A_p \mod \mathcal{C}_p$ for any A, the claim follows. $\square$

We now propagate this calculation using suspension.

Thm For any odd $n>1$ and prime p, the double suspension

$$\pi_i(S^n) \longrightarrow \pi_{i+2}(S^{n+2})$$

is an isomorphism mod $\mathcal{C}_p$ for $i<p(n+1)-3$.

Lemma Let C be an Abelian group. Consider the following conditions.

(1) $C \in \mathcal{C}_p$.

(2) C is torsion and multiplication by p is injective on C.

(3) Multiplication by p is bijective on C.

(4) $C \otimes \mathbb{F}_p = 0 = \mathrm{Tor}(C, \mathbb{F}_p)$.

(5) $C \otimes \mathbb{F}_p = 0$.

Then (1) $\iff$ (2) $\implies$ (3) $\iff$ (4) $\implies$ (5) and all are equivalent if C is finitely generated.

Ex The localization of $\mathbb{Z}$ at the multiplicative set $\{p^n \mid n > 0\}$ satisfies (3) but not (2). Its quotient by $\mathbb{Z}$ satisfies (5) but not (4).

Proof (1) $\iff$ (2) is the definition of $\mathcal{C}_p$.

Assuming (2), Fermat's little theorem shows that C is p-divisible, implying (3). Since $C \otimes \mathbb{F}_p \cong C/pC$ and $\mathrm{Tor}(C, \mathbb{F}_p) = \ker(C \xrightarrow{p} C)$, (3) $\iff$ (4). Finally, if C is finitely generated and not torsion, then $C \cong C' \oplus \mathbb{Z}$, so C is not p-divisible, implying (2), and the

Implication (5) $\Rightarrow$ (4) follows from the classification of finitely generated Abelian groups — if $C \otimes \mathbb{F}_p = 0$, then C has no p-power torsion summands, so $\mathrm{Tor}(C, \mathbb{F}_p) = 0$. $\square$

<u>Lemma</u> The following are equivalent for a homomorphism $\varphi : A \to B$ between finitely generated Abelian groups

(1) φ e_p-isomorphism.

(2) $\ker \varphi \otimes \mathbb{F}_p$, $\operatorname{coker} \varphi \otimes \mathbb{F}_p$ trivial.

(3) $\ker(\varphi \otimes \mathbb{F}_p)$, $\operatorname{coker}(\varphi \otimes \mathbb{F}_p)$ trivial and $\mathrm{Tor}(A, \mathbb{F}_p) \to \mathrm{Tor}(A/\ker \varphi, \mathbb{F}_p)$ isomorphism.

(4) $\varphi\otimes\mathbb{F}_p$ and $\mathrm{Tor}(\varphi,\mathbb{F}_p)$ isomorphisms.

Example The quotient map $\mathbb{Z}/p^2\mathbb{Z}\to\mathbb{Z}/p\mathbb{Z}$ satisfies the first half of (4) but is not a e_p-isomorphism.

Proof (1)$\Longleftrightarrow$(2) follows from the lemma using finite generation (and Noetherianity). For (2)$\Longleftrightarrow$(3), we have $\mathrm{coker}\,\varphi\otimes\mathbb{F}_p\cong\mathrm{coker}(\varphi\otimes\mathbb{F}_p)$ by right exactness, and the exact sequence

$$\begin{tikzcd}
\mathrm{Tor}(A,\mathbb{F}_p) \arrow[r] & \mathrm{Tor}(A/\ker\varphi,\mathbb{F}_p) \arrow[r] & \ker\varphi\otimes\mathbb{F}_p \arrow[r] \arrow[dr] & A\otimes\mathbb{F}_p \\
\mathrm{Tor}(\ker\varphi,\mathbb{F}_p) \arrow[u] & & & \ker(\varphi\otimes\mathbb{F}_p) \arrow[u]
\end{tikzcd}$$

(we use finite generation and the lemma to conclude that $\mathrm{Tor}(\ker\varphi, \mathbb{F}_p) = 0$. For (3) $\Longleftrightarrow$ (4), consider the exact sequence

$$0 \to \mathrm{Tor}(A/\ker\varphi, \mathbb{F}_p) \to \mathrm{Tor}(B, \mathbb{F}_p) \to \mathrm{Tor}(\operatorname{coker}\varphi, \mathbb{F}_p).$$

In either situation, $\operatorname{coker}\varphi \otimes \mathbb{F}_p = 0$, so the righthand term vanishes by finite generation and the lemma. $\square$

<u>Cor</u> For a map $f: X \to Y$ between simply connected spaces of finite type, consider the following.

(1_r) f induces a $\mathcal{C}_p$-isomorphism on H_i for every $i \leq r$

(2r) f induces an isomorphism on $H_i(-;\mathbb{F}_p)$ for every $i \le r$.

Then $(2r) \Rightarrow (1(r-1)) \Rightarrow (2(r-1))$.

<u>Proof</u> Apply UCT, induction, and the five lemma. $\square$