Last time

- Proof of rank theorem
- $\pi_i(S^3)_p$, $i \leq 2p$
- Mod $\mathcal{C}_p$ isomorphisms

Thm For any odd $n>1$ and prime p, the double suspension

$$\pi_i(S^n) \longrightarrow \pi_{i+2}(S^{n+2})$$

is an isomorphism mod $\mathcal{C}_p$ for $i<p(n+1)-3$.

From last time, we know that mod $\mathcal{C}_p$ isomorphisms can be detected by mod p homology.

More specifically, given $f: X \to Y$ (simply connected, finite type),

$$\begin{matrix} f_* \text{ iso on} \\ H_i(-;\mathbb{F}_p),\ i \le r \end{matrix} \Longrightarrow \begin{matrix} f_*\ e_p\text{-iso} \\ \text{on } H_i,\ i<r \end{matrix} \Longrightarrow \begin{matrix} f_* \text{ iso on} \\ H_i(-;\mathbb{F}_p),\ i<r \end{matrix}$$

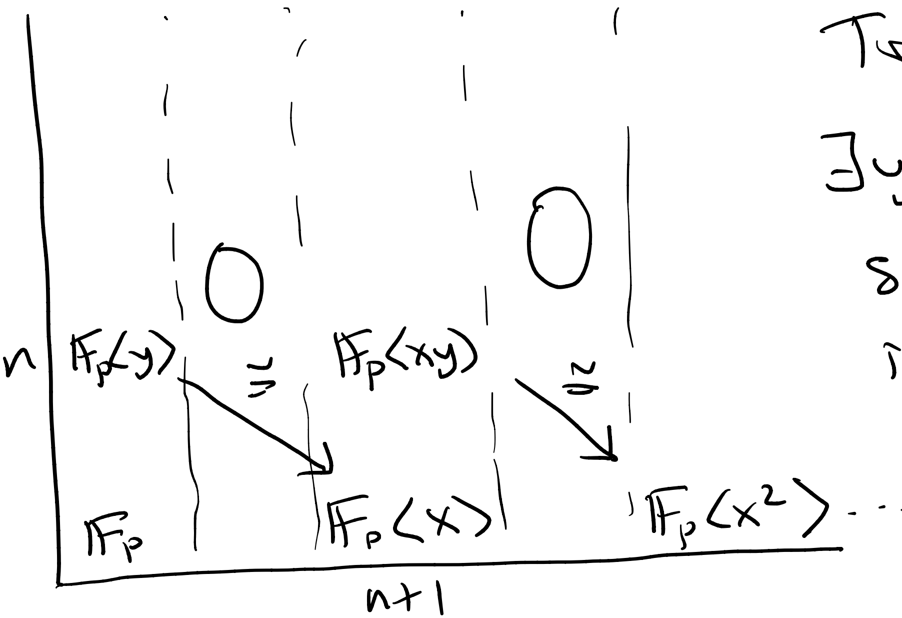
<u>Lemma</u> For odd $n>1$ and prime p, we have $H_k(\Omega^2 S^{n+2}_1; \mathbb{F}_p) = 0$ for $n<k<p(n+1)-1$.

<u>Proof</u> It follows easily from our previous calculation of $H^*(\Omega S^{n+2})$ that, for $i<p(n+1)$,

$$H^i(\Omega S^{n+2}; \mathbb{F}_p) \cong \begin{cases} \mathbb{F}_p\langle x^k \rangle & i = k(n+1) \\ 0 & \text{otherwise.} \end{cases}$$

Consider the SSS in $\mathbb{F}_p$ cohomology for

$$\Omega^2 S^{n+2} \to \overset{\simeq pt}{P\Omega S^{n+2}} \to \Omega S^{n+2}.$$



To avoid contradiction, $\exists y \in H^n(\Omega^2 S^{n+2}; \mathbb{F}_p)$ such that $d_{n+1} y = x$ is an isomorphism. Applying Leibniz, we see that

$d_{n+1} x^k y = x^{k+1}$. Thus, the next nonzero differential exiting the 0^{th} column has target in degree $p(n+1)$, hence sovere in degree $p(n+1)-1$. The claim follows from UCT.

$\square$

Proof of thm The double suspension of the lemma is induced by the map

$$S^n \xrightarrow{\eta} \Omega S^{n+1} \xrightarrow{\Omega\eta} \Omega^2 S^{n+1},$$

which induces an isomorphism on π_n, hence on H_n by Hurewicz, hence an isomorphism on $H_k(-;\mathbb{F}_p)$ for $k < p(n+1)-1$ by the lemma, hence a $\mathcal{C}_p$-isomorphism on H_n for $k < p(n+1)-2$ by the corollary, hence on π_k for $k < p(n+1)-3$ by relative Hurewicz mod $\mathcal{C}_p$ (we have used that $\Omega^2 S^{n+2}$ is of finite type, since ΩS^{n+2} is). $\square$

<u>Cor</u> For odd $n > 1$, p prime, and $i < 4p+n-6$

$$\pi_i(S^n)_p \cong \pi_{i-n+3}(S^3)_p$$

<u>Proof</u> Induction on n with base case $n=3$.

For $n \geq 5$, the theorem grants that

$$\pi_i(S^n)_p \cong \pi_{i-2}(S^{n-2})_p$$

for $i-2 < p(n-1)-3$, so we must check

the inequality $4p+n-6 \leq p(n-1)-1$. Rearranging and factoring gives $(p-1)(n-5) \geq 0$.

$\square$

<u>Cor</u> For odd $n>1$, p prime, $\pi_{n+m}(S^n)_p = 0$ for $0<m<2p-3$, and $\pi_{n+2p-3}(S^n)_p \cong \mathbb{Z}/p\mathbb{Z}$.

<u>Cor</u> For $i \leq 2p-3$,

$$(\pi_i^s)_p \cong \begin{cases} \mathbb{Z}/p\mathbb{Z} & i=2p-3 \\ 0 & \text{otherwise.} \end{cases}$$

To go further in calculating homotopy groups of spheres, we recognize the key map $S^n \to K(\mathbb{Z}, n)$ as the first stage of the Postnikov tower for S^n:

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & X_2 \arrow[d] & K(\pi_{n+2}, n+2) \arrow[l] \\
 & X_1 \arrow[d] & K(\pi_{n+1}, n+1) \arrow[l] \\
S^n \arrow[r] \arrow[ur] \arrow[uur] \arrow[uuur] & X_0 = K(\mathbb{Z}, n) &
\end{tikzcd}
$$

$$
\pi_i(X_k) = \begin{cases} \pi_i(S^n) & i \le n+k \\ 0 & \text{otherwise} \end{cases}
$$

It turns out that these fiber sequences fit into a commuting diagram

$$\begin{tikzcd}
K(\pi_{n+k}, n+k) \arrow[r, equal] \arrow[d] & K(\pi_{n+k}, n+k) \arrow[d] \\
X_k \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \text{contractible} \arrow[d] \\
X_{k-1} \arrow[r] & K(\pi_{n+k}, n+k+1)
\end{tikzcd},$$

so we can hope to exploit naturality in inductive calculations. This fact relies on a surprising double role played by Eilenberg-MacLane spaces.

Thm There is a canonical natural isomorphism for pointed spaces X

$$\tilde{H}^n(X;G) \cong [X, K(G,n)].$$

Rk In categorical terms, the theorem asserts that the functor of cohomology is <u>representable</u>.

Evidence $\tilde{H}^n(S^m;G) \cong \pi_m(K(G,n)) = [S^m, K(G,n)]$

Strategy (1) Show $[X, K(G,n)]$ is like cohomology.

(2) Show that anything like cohomology actually is.

Def A (reduced) <u>cohomology theory</u> (for pointed CW complexes) is the data of

(1) Abelian groups $T^n(X)$ for every $n \in \mathbb{Z}$ and pointed CW complex X,

(2) homomorphisms $f^*: T^n(Y) \to T^n(X)$ for every $n \in \mathbb{Z}$ and pointed map $f: X \to Y$,

(3) homomorphisms $\delta: T^n(A) \to T^{n+1}(X/A)$ for every $n \in \mathbb{Z}$ and pointed CW pair (X, A),

subject to the following axioms:

(functoriality) $id^* = id$, $(f \circ g)^* = g^* \circ f^*$, and

$$\delta^* \circ f = f \circ \delta^* ;$$

(homotopy invariance) if $f \simeq g$, then $f^* = g^*$;

(additivity) the canonical homomorphism

$$T^n(\bigvee_\alpha X_\alpha) \longrightarrow \prod_\alpha T^n(X_\alpha)$$

is an isomorphism;

(exactness) the following sequence is exact:

$$\cdots \to T^n(A) \xrightarrow{\delta} T^{n+1}(X/A) \xrightarrow{q^*} T^{n+1}(X) \xrightarrow{i^*} T^{n+1}(A) \to \cdots .$$

A map of cohomology theories is a collection of homomorphisms $\varphi : T_1^n(X) \longrightarrow T_2^n(X)$ such that $\varphi \circ \delta = \delta \circ \varphi$ and $\varphi \circ f^* = f^* \circ \varphi$.

Thm A map of cohomology theories $\varphi: T_1^* \longrightarrow T_2^*$ is an isomorphism iff the map $\varphi: T_1^n(S^0) \longrightarrow T_1^n(S^0)$ is an isomorphism for every $n \in \mathbb{Z}$.

Rmk One can show that, if $T^n(S^0) = 0$ for $n \neq 0$, then the isomorphism

$$T^0(S^0) \cong H^0(S^0; T^0(S^0))$$

extends to a map (hence isomorphism) of cohomology theories. So ordinary cohomology is the unique cohomology theory satisfying the "dimension" axiom $T^{\neq 0}(S^0) = 0$.

Proof For simplicity, we restrict our attention to finite dimensional complexes. The general case reduces to this case via the "Milnor exact sequence".

First, by additivity, we have $T_i^*(pt) = 0$ (consider wedges and products over empty indexing sets). Hence $T_i^*(D^n) = 0$ by homotopy invariance, and exactness gives the suspension isomorphism $T_i^m(S^n) \xrightarrow[s]{\cong} T_i^{m+1}(S^{n+1})$. The diagram

$$\begin{tikzcd} T_1^m(S^n) \arrow[r, "\delta"] \arrow[d, "\varphi"'] & T_1^{m+1}(S^{n+1}) \arrow[d, "\varphi"] \\ T_2^m(S^n) \arrow[r, "\delta"] & T_2^{m+1}(S^{n+1}) \end{tikzcd}$$

commutes by functoriality; thus, since φ is an isomorphism on S^0, it is so on S^n for every $n \geq 0$ by induction.

Let X be an n-dimensional pointed CW complex. If $n=0$, then $X \cong \bigvee_{X \setminus x_0} S^0$, so φ is an isomorphism on X by additivity. Otherwise, we have the commuting diagram with exact rows

$$
\begin{tikzcd}
T_1^{m-1}(X_{n-1}) \arrow[r] \arrow[d, "\varphi"', "\cong"] & T_1^m\left(\bigvee_\alpha S^n\right) \arrow[r] \arrow[d, "\varphi"', "\cong"] & T_1^m(X_n) \arrow[r] \arrow[d, "\varphi"'] & T_1^m(X_{n-1}) \arrow[d, "\varphi"', "\cong"] \\
T_2^{m-1}(X_{n-1}) \arrow[r] & T_2^m\left(\bigvee_\alpha S^n\right) \arrow[r] & T_2^m(X_n) \arrow[r] & T_1^m(X_{n-1})
\end{tikzcd}
$$

where α ranges over the set of n-cells of X. The indicated maps are isomorphisms by induction, additivity, and the case of a sphere. Since $X_n = X$, the claim follows. $\square$