Last time

- $(\pi_i^s)_p$ for $i \leq 2p-3$
- Cohomology theories

Representability theorem $\tilde{H}^n(-;G) \cong [-, K(G,n)]$

Before proving this theorem, we observe some important consequences for the structure of cohomology.

Def A **cohomology operation** of type $(G, n; H, m)$ is a natural transformation

$$\theta : H^n(-;G) \longrightarrow H^m(-;H)),$$

i.e., a collection of functions θ_X for pointed CW complexes X such that every diagram of the following form commutes:

$$\begin{tikzcd} H^n(X;G) \arrow[r, "\theta_X"] & H^m(X;H) \\ H^n(Y;G) \arrow[u, "f^*"] \arrow[r, "\theta_Y"] & H^m(Y;H) \arrow[u, "f^*"'] \end{tikzcd}$$

Warning The functions θ_X need not be homomorphisms.

Example The function $\alpha \longmapsto \alpha^2$ is a cohomology operation of type $(\mathbb{Z}, n; \mathbb{Z}, 2n)$ for each n.

Example Applying $\mathrm{Hom}(C_*(X), -)$ to a short exact sequence $0 \to A \to B \to C \to 0$ leads to a long exact sequence

$$\cdots \to H^n(X;A) \to H^n(X;B) \to H^n(X;C) \xrightarrow{\delta} H^{n+1}(X;A) \to \cdots,$$

whose connecting homomorphism is a

cohomology operation of type $(C, n; A, n+1)$ called a Bockstein homomorphism. Prominent examples are the sequences

$0 \to \mathbb{Z}/p\mathbb{Z} \xrightarrow{p} \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$ & $0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$

Note that cohomology operations of fixed type form an Abelian group.

Thm (Serre) There is a canonical isomorphism

$$\left\{ \begin{array}{c} \text{cohomology} \\ \text{operations of} \\ \text{type } (G, n; H, m) \end{array} \right\} \xrightarrow{\cong} H^m(K(G,n); H).$$

This result follows formally from representability.

Indeed, it is a special case of the (trivial) Yoneda lemma from category theory.

<u>Skepticism</u> How is $[X, K(G,n)]$ even a group?

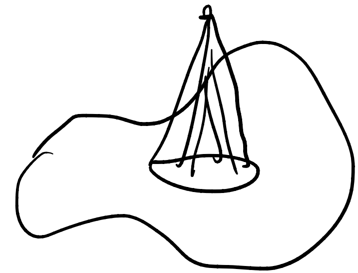
<u>Observation</u> $K(G,n) \simeq \Omega^r K(G,n+r)$ for G Abelian.

<u>Exercise</u> $[X, \Omega^r Y]$ is canonically a group, Abelian if $r>1$.

<u>Idea</u> Connecting homomorphisms are the same information as suspension isomorphisms.

Def The <u>mapping cone</u> of $f: X \to Y$ is

$$C_f = M_f / X \times \{0\}.$$



If f is a pointed map, the <u>reduced</u> mapping cone is

$$\tilde{C}_f = C_f / \{x_0\} \times [0,1].$$

Prop For any pointed map $f: X \to Y$ and pointed space Z, the sequence

$$[\tilde{C}_f, Z] \longrightarrow [Y, Z] \longrightarrow [X, Z]$$

is exact.

Proof Exercise.

$\square$

Thus, applying $[-, Z]$ to the "cofiber" or "Puppe" sequence

$$X \xrightarrow{f} Y \xrightarrow{g} \tilde{C}_f \xrightarrow{h} \tilde{C}_g \longrightarrow \tilde{C}_h \longrightarrow \cdots$$

yields an exact sequence. When f is the inclusion of a subspace, the mapping cone serves as a well-behaved proxy for the quotient.

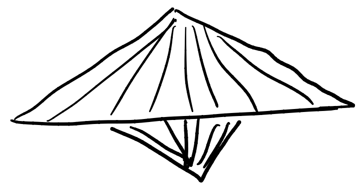
Lemma If $\iota : A \subseteq X$ is a good pair, then

(1) $C_\iota \xrightarrow{\simeq} X/A$.

If (A, x_0) is also a good pair, then

(2) $C_\iota \xrightarrow{\simeq} \tilde{C}_\iota$,

(3) $\tilde{C}(X \to \tilde{C}_\iota) \xrightarrow{\simeq} \Sigma A$



Moreover,

(4) $C_{\Sigma f} \simeq \Sigma C_f$.

Proof Exercise. $\square$

Cor With the same assumptions, there is a canonical exact sequence

$$\cdots \to [\Sigma A, Z] \xrightarrow{\delta} [X/A, Z] \longrightarrow [X, Z] \longrightarrow [A, Z],$$

where δ is given by restriction along $X/A \xleftarrow{\sim} C_i \longrightarrow \Sigma A$.

Before continuing, we observe that ordinary connecting homomorphisms arise in precisely this way.

Lemma The following diagram and the dual diagram in cohomology commute:

$$
\begin{tikzcd}
H_n(X,A) \arrow[r, "\cong"] \arrow[d, "\delta"'] & H_n(C_\iota, CA) & H_n(C_\iota) \arrow[l, "\cong"'] \arrow[d] \\
H_{n-1}(A) & H_n(CA,A) \arrow[l, "-\delta"'] \arrow[r, "\cong"] & H_n(C_\iota, X).
\end{tikzcd}
$$

In particular if (X,A) is a good pair, then so does

$$
\begin{tikzcd}
\tilde{H}_n(X/A) \arrow[d, "-\delta"'] \arrow[dr, "(X/A \to \Sigma A)_*"] & \\
\tilde{H}_{n-1}(A) \arrow[r, equal, "\cong"] & \tilde{H}_n(\Sigma A)
\end{tikzcd}
$$

Proof Given a cycle $c \in C_n(X,A)$, choose $\tilde{c} \in C_n(CA)$ such that $\partial \tilde{c} = \partial c$.

$$
\begin{tikzcd}
{[c]} \arrow[rr, mapsto] \arrow[dd, mapsto] & & {[c]} \arrow[rr, mapsto] & & {[c-\tilde{c}]} \arrow[ddd, mapsto] \\
& H_n(X,A) \arrow[r, "\cong"] \arrow[d, "\delta"'] & H_n(C_\iota, CA) & H_n(C_\iota) \arrow[l, "\cong"'] \arrow[d] & \\
{[\partial c]} & H_{n-1}(A) & H_n(CA,A) \arrow[l, "\delta"'] \arrow[r, "\cong"] & H_n(C_\iota, X) & \\
& -[\partial \tilde{c}] & -[\tilde{c}] \arrow[l, mapsto] & & -[\tilde{c}] \arrow[ll, mapsto]
\end{tikzcd}
$$

$\square$

It is now easy to check that the following data defines a cohomology theory T^*:

$$T^n(X) = [X, K(G,n)]$$

$$f^*[g] = [g \circ f],$$

and connecting homomorphisms

$$-\delta : [A, K(G,n)] \cong [A, \Omega K(G,n+1)] \cong [\Sigma A, K(G,n+1)] \longrightarrow [X/A, K(G,n+1)].$$

Thus, it remains to construct a map $\varphi : T^* \longrightarrow \tilde{H}^*(-;G)$ inducing an isomorphism on S^0.

<u>Construction</u> Consider our preferred $K(G,n)$ with n-skeleton $\bigvee_I S^n$. We have $\tilde{C}^{CW}_n(K(G,n)) \cong \bigoplus_I \mathbb{Z}$, and we define $\lambda_n : \tilde{C}^{CW}_n(K(G,n)) \longrightarrow G$ by sending 1_i to the $i^{\underline{th}}$ generator of G

<u>Exercise</u> $\lambda_n \in \tilde{C}^n_{CW}(K(G,n); G)$ is a cocycle.

<u>Exercise</u> The composite

$$G \xrightarrow{\cong} \pi_n(K(G,n)) \xrightarrow{h} H_n(K(G,n)) \xrightarrow{\langle -, \lambda_n \rangle} G$$

is the identity, and this uniquely specifies λ_n.

Define homomorphisms

$$\begin{aligned} \varphi_n : [X, K(G,n)] &\longrightarrow \tilde{H}^n(X;G) \\ [f] &\longmapsto f^*\lambda_n . \end{aligned}$$

We will show that the φ_n form a map of cohomology theories.

<u>Lemma 2</u> For any X, the following diagram commutes:

$$\begin{tikzcd}
{[X, K(G,n)]} \arrow[r, "\varphi_n"] \arrow[d] & \tilde{H}^n(X;G) \arrow[d] \\
{[\Sigma X, K(G,n+1)]} \arrow[r, "\varphi_{n+1}"] & \tilde{H}^{n+1}(\Sigma X;G).
\end{tikzcd}$$

Abbreviate?

<u>Proof of theorem</u> It is an easy exercise to show that φ_n is an isomorphism on S^0 for every n, so it suffices to show that the various φ_n

define a map of cohomology theories. The only nontrivial claim is commutativity of the middle square in the diagram

$$\begin{tikzcd}
{[\Sigma A, K(G,n+1)]} \arrow[dr, "(X/A \to \Sigma A)^*"'] \arrow[rrr, bend left, "\varphi_{n+1}", "1"'] \arrow[r, phantom, "3", near end, shift right=6ex] & {[A, K(G,n)]} \arrow[l, "\cong"'] \arrow[r, "\varphi_n"] \arrow[d, "-\delta"] \arrow[dr, phantom, "?"] & \tilde{H}^n(A;G) \arrow[r, "\cong"] \arrow[d, "-\delta"] \arrow[r, phantom, "2", shift right=6ex] & \tilde{H}^{n+1}(\Sigma A;G) \arrow[dl, "(X/A \to \Sigma A)^*"] \\
& {[X/A, K(G,n+1)]} \arrow[r, "\varphi_{n+1}"] & \tilde{H}^{n+1}(X/A;G) &
\end{tikzcd}$$

.

But diagrams 1, 2, and 3 commute by Lemmas 1 and 2 and by definition, and the claim follows. $\square$

Lemma 3 The homomorphism

$$[K(G,n), K(G,n)] \longrightarrow \mathrm{Hom}(G,G)$$
$$[f] \longmapsto f_*$$

is an isomorphism.

Proof Essentially the same argument as showing that any two $K(G,n)$ spaces are weakly equivalent. $\square$

Abbreviate?

Proof of Lemma 2 Fix $f: X \to K(G,n)$, and consider the commuting diagram

$$
\begin{tikzcd}
{[\bigvee_I S^n, K(G,n)]} \arrow[rrrrr, "\varphi_n"] \arrow[ddddd, "\cong"'] & & & & & \tilde{H}^n(\bigvee_I S^n; G) \arrow[ddddd, "\cong"] \\
& {[K(G,n), K(G,n)]} \arrow[ul] \arrow[rrr, "\varphi_n"] \arrow[dr, "f^*"'] \arrow[ddd, "\cong"'] & & & \tilde{H}^n(K(G,n); G) \arrow[ur] \arrow[dl, "f^*"] \arrow[ddd, "\cong"] & \\
& & {[X, K(G,n)]} \arrow[r, "\varphi_n"] \arrow[d, "\cong"'] & \tilde{H}^n(X; G) \arrow[d, "\cong"] & & \\
& & {[\Sigma X, K(G,n+1)]} \arrow[r, "\varphi_{n+1}"] & \tilde{H}^{n+1}(\Sigma X; G) & & \\
& {[\Sigma K(G,n), K(G,n+1)]} \arrow[ur, "\Sigma f^*"] \arrow[rrr, "\varphi_{n+1}"] \arrow[dl] & & & \tilde{H}^{n+1}(\Sigma K(G,n); G) \arrow[ul, "\Sigma f^*"'] \arrow[dr] & \\
{[\bigvee_I S^{n+1}, K(G,n+1)]} \arrow[rrrrr, "\varphi_{n+1}"] & & & & & \tilde{H}^{n+1}(\bigvee_I S^{n+1}; G)
\end{tikzcd}
$$

Since $f^*[\mathrm{id}] = [f]$, commutativity of the inner diagram for $[f]$ follows from commutativity of the middle diagram for $[\mathrm{id}]$. The outward facing diagonal arrows are injective by Lemma 3, so it suffices to establish commutativity of the outer diagram. Thus, it suffices to establish the case $X = S^n$, which is an exercise

$\square$