PROBLEM LIST 3
TOPOLOGY 3, SPRING 2024

Problem 1. Consider the following commuting diagram of homomorphisms among Abelian groups:

$$\begin{tikzcd}
\cdots \arrow[rr, red] & & A \arrow[rr, green] \arrow[dr, red] & & B \arrow[rr, blue] \arrow[dr, green] & & C \arrow[rr, yellow] \arrow[dr, blue] & & D \arrow[rr, red] \arrow[dr, yellow] & & \cdots \\
& \cdots \arrow[ur, green] \arrow[dr, yellow] & & E \arrow[ur, blue] \arrow[dr, red] & & F \arrow[ur, yellow] \arrow[dr, green] & & G \arrow[ur, red] \arrow[dr, blue] & & \cdots & \\
\cdots \arrow[rr, blue] & & H \arrow[rr, yellow] \arrow[ur, blue] & & I \arrow[rr, red] \arrow[ur, yellow] & & J \arrow[rr, green] \arrow[ur, red] & & K \arrow[rr, blue] \arrow[ur, green] & & \cdots
\end{tikzcd}$$

Suppose that the red, green, and blue sequences are exact and that each consecutive pair of homomorphisms in the yellow sequence composes to zero. Show that the yellow sequence is exact (it suffices to check exactness at I, F, and C).

Definition 1. Let X be a CW complex, and write $i_n : X_n \to X_{n+1}$ for the inclusion. The *telescope* on X is

$$\mathrm{Tel}(X) = \frac{\coprod_{n\geq 0} X_n \times [0,1]}{(x,1) \sim (i_n(x),0)}.$$

Problem 2. Let X be a CW complex.

(a) Draw a picture of $\mathrm{Tel}(X)$.
(b) Explain why $\mathrm{Tel}(X)$ is a CW complex (no need for a complete argument).
(c) Can you define a comparison map $\mathrm{Tel}(X) \to X$?
(d) Suppose that X is finite dimensional. Show that your map from (c) is a homotopy equivalence.
(e) Show that your map from (c) is a homotopy equivalence in general (hint: Whitehead).

Problem 3. Let T be a cohomology theory for CW complexes (unreduced, without basepoints).

(a) We didn't define this concept in class. Explain how to modify the axioms we gave to make a reasonable definition.
(b) Let X be a CW complex and $U, V \subseteq X$ subcomplexes with $X = U \cup V$. Show that there is a Mayer–Vietoris exact sequence

$$\cdots \to T^n(X) \to T^n(U) \oplus T^n(V) \to T^n(U \cap V) \xrightarrow{(\star)} T^{n+1}(X) \to \cdots$$

(hint: Problem 1). Note that the construction of the starred arrow is part of the problem.

(c) Show that there is an exact sequence

$$\cdots \to T^n(X) \to \prod_{p\geq 0} T^n(X_p) \to \prod_{p\geq 0} T^n(X_p) \to T^{n+1}(X) \to \cdots$$

(hint: consider $\coprod X_{2m} \times [0,1]$ and $\coprod X_{2m+1} \times [0,1]$).

(d) Show that a map of cohomology theories is an isomorphism if and only if it is an isomorphism on all finite dimensional CW complexes.
(e) Explain how to use an unreduced cohomology theory to define a reduced cohomology theory and vice versa. Explain how this problem completes the proof of a theorem from class.

The kernel and cokernel of the middle homomorphism above are purely algebraic constructions called $\varprojlim$ and $\varprojlim^1$, respectively. The resulting short exact sequence

$$0 \to \varprojlim{}^1 T^{n-1}(X_n) \to T^n(X) \to \varprojlim T^n(X_n) \to 0$$

is called the *Milnor exact sequence.*

Definition 2. A pointed space X is *well-pointed* if (X, x_0) is a good pair.

Problem 4. In this problem, $\iota : A \to X$ is the inclusion of a closed subspace and $x_0 \in X$ is any basepoint.

(a) Show that, if (X, A) is a good pair, then the quotient map $C_\iota \to X/A$ is a homotopy equivalence (here C_ι is the mapping cone as defined in class).
(b) Give a picture proof that $(X \times [0,1], X \times \{0,1\} \cup \{x_0\} \times [0,1])$ is a good pair if X is well-pointed.
(c) Use (b) to show that $(SX, \{x_0\} \times [0,1])$ is a good pair if X is well-pointed.
(d) Use (a) and (c) to show that the quotient map $SX \to \Sigma X$ is a homotopy equivalence if X is well-pointed (don't work too hard).
(e) Use (c) to show that ΣX is well-pointed if X is so.